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**Different Hagedorn temperatures for mesons  
and baryons from experimental mass spectra,  
compound hadrons, and combinatorial saturation**

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**Abstract**

We analyze the light-flavor particle mass spectra and show that in the region up to  $\simeq 1.8\text{GeV}$  the Hagedorn temperature for baryons is about 30% smaller than for mesons, reflecting the fact that the number of baryon states grows more rapidly with the mass. We also show that the spectra are well reproduced in a model where hadrons are compound objects of quanta, whose available number increases with mass. The rapid growth of number of hadronic states is a combinatorial effect. We also point out that an upper limit on the excitation energy of these quanta results in a maximum number of hadron states that can be formed. According to this combinatorial saturation, no more light-flavor hadron resonances exist above a certain mass.

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In 1965 Hagedorn [1] postulated that for large masses  $m$  the spectrum of hadrons grows exponentially,  $\rho(m) \sim \exp(m/T_H)$ , where  $T_H$ , the Hagedorn temperature, is a scale parameter. The hypothesis was based on the observation that at some point a further increase of energy in  $pp$  and  $p\bar{p}$  collisions no longer raises the temperature of the formed fireball, but results in more and more particles being produced. Thus, there is a maximum temperature that a hadronic system can achieve. The Statistical Bootstrap Model [2,3] predicted that asymptotic behavior, namely  $\rho(m) \simeq cm^a \exp(m/T_H)$ , where  $a$  is a negative power ( $a \leq -5/2$  [2], or  $a \leq -3$  [3]). The parameters of the asymptotic spectrum are to be determined by comparing to the experimental mass spectra. The fits of the sixties had a rather poor spectrum to their disposal, sufficiently dense only in the range of masses up to 1GeV, and sparse above. Still, Hagedorn and Ranft [4] were able to fit the function

$$\rho_{HR}(m) = \frac{c}{(m^2 + m_0^2)^{5/4}} \exp(m/T_H), \quad (1)$$

with  $m_0 = 0.5\text{GeV}$ , and got  $T_H = 160\text{MeV}$ . Frautschi [5] obtained similar values for the Hagedorn temperature. Ever since it has been believed that there is one universal scale in the asymptotic spectrum of hadrons, and its value is about 160MeV. Meanwhile, the particle tables became more complete and longer, with a total of 3182 light-flavor states (counting mesons, baryons, and antibaryons with their spin and isospin degeneracies) compared to 1432 used in Ref. [4]. This allows for a much more stringent verification of the Hagedorn hypothesis. We have done this, with surprising results.

We have used the 1998 edition of the Particle Data Group tables [6]. Rather than comparing the experimental and theoretical spectra, we compare their *cumulants*: the number of states with the energy less than  $m$ ,

$$N_{\text{exp}}(m) = \sum_i g_i \Theta(m - m_i), \quad (2)$$

and

$$N_{\text{theor}}(m) = \int_0^m \rho_{\text{theor}}(m') dm' \quad (3)$$

where  $\Theta$  is the step function,  $m_i$  is the mass of the resonance, and  $g_i$  is its degeneracy. Obviously,  $dN(m)/dm = \rho(m)$ . Although the information content is the same whether one uses  $\rho(m)$  directly, or  $N(m)$ , the use of cumulants is advantageous, since that way one avoids the nuisance of producing histograms of  $\rho_{\text{exp}}$ . Our results are presented in Fig. 1, where we show the  $\log_{10} N(m)$  separately for mesons and baryons. We can clearly see that up to  $m \simeq 1.8\text{GeV}$  the data line-up along (almost) straight lines.

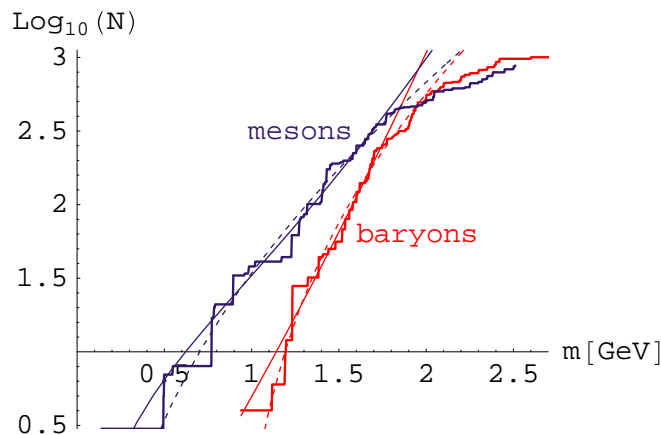


FIG. 1. Cumulants of meson and baryon spectra, Hagedorn fits (solid lines), and Compound Hadron Model fits (dashed lines), plotted as functions of mass.

The solid lines are the least-square fits to the data according to Eq. (1) over the range up to 1.8GeV, skipping the lightest particle in the set. The behavior of Fig.1 is compatible with the Hagedorn hypothesis.

However, the slopes in Fig. 1 are different for mesons and baryons, which means *different Hagedorn temperatures for mesons and baryons* for masses in the region up to 1.8GeV. This is the key observation of this paper. Applying the form (1), with  $m_0 = 500\text{MeV}$ , we find the best fit for the mesons with  $T_{\text{mes}} = 197\text{MeV}$ , and for the baryons with  $T_{\text{bar}} = 141\text{MeV}$ . These temperatures differ considerably from each other, reflecting the fact that the *baryon spectrum grows much more rapidly*.

A question arises if it were possible that meson states are missed in experiments much more frequently than baryons, such that the effect of Fig. 1 is spurious. The numbers show this is highly unlikely. In order to make the meson line parallel to the baryon line we would have to aggregate as much as 500 additional states up to  $m = 1.8\text{GeV}$ , more than the present number of 407 states at this point. Thus, we reject the scenario of missing meson states. We also remark that discoveries of some extra states can only increase the slopes in Fig. 1, thus lowering the presently-fitted values of  $T_{\text{mes}}$  and  $T_{\text{bar}}$ . Another question is whether we have reached the asymptotics with  $m \sim 1.8\text{GeV}$ . This question can only be addressed within a sufficiently accurate theory or model. It should be stressed that the function multiplying the exponent has significance for the fitted values of  $T_{\text{mes}}$  and  $T_{\text{bar}}$ . For instance, with  $m_0 = 1\text{GeV}$  we obtain 228MeV and 152MeV for mesons and baryons, respectively. Using different values of  $m_0$  for mesons and baryons may bring the two temperatures closer to each other. Anyway, in the range of  $m$  up to  $\sim 1.8\text{GeV}$  we *effectively* have two distinct Hagedorn temperatures for mesons and baryons, as seen in Fig. 1.

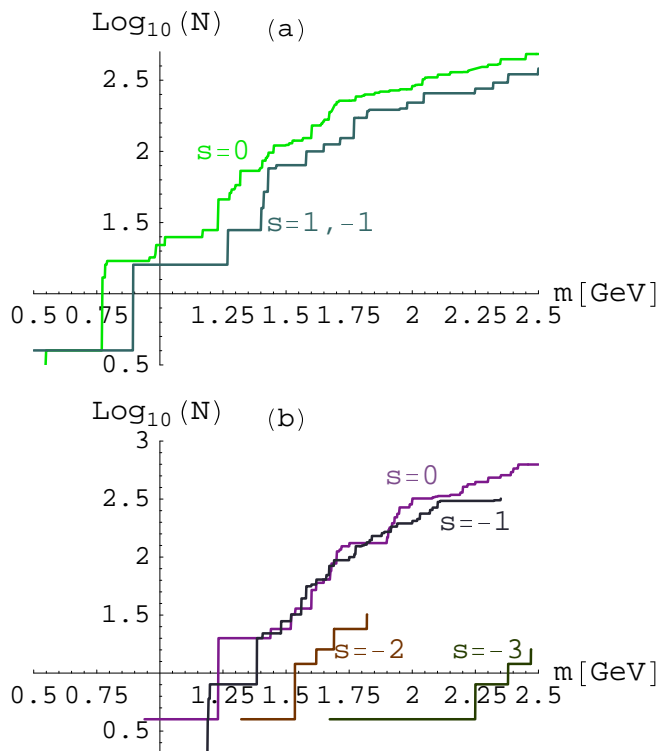


FIG. 2. Cumulants of meson (a) and baryon (b) spectra with selected strangeness, plotted as functions of mass.

In Fig. 2 we compare the cumulants of spectra for mesons and baryons of definite strangeness. We can see the *universality of slopes* for both mesons and baryons.

The theoretical challenge is to explain the different behavior of mesonic and baryonic spectra. Historically, the Dual String Model [7] produced the exponentially-growing mass

spectra. In the framework of the Veneziano model [8] Huan and Weinberg [9] derived the following asymptotic formula:

$$\rho_V(m) \sim 2\alpha' m P_D(n), \quad (4)$$

$$P_D(n) = \sqrt{\frac{1}{2n}} \left(\frac{D}{24n}\right)^{\frac{D+1}{4}} \exp\left(2\pi\sqrt{\frac{Dn}{6}}\right), \quad (5)$$

$$n = \alpha_0 + \alpha' m^2, \quad (6)$$

where  $\alpha'$  is the slope of the Regge trajectories,  $\alpha_0$  is the intercept (averaged over various trajectories), and  $D$  is the number of dimensions, usually assumed to be 4 [9], although higher values have also been attempted [10]. In the strict sense, the dual string models are constructed for mesons, and their application to baryons has a more phenomenological character. Since  $\alpha' \approx 1\text{GeV}^{-2}$  is universal for all hadrons, the difference between mesons and baryons could only come from different values of  $\alpha_0$ , and possibly from  $D$ . We have tried all reasonable values for these parameters, and concluded that they result in very similar curves. The function multiplying the exponent in Eq. (4) has significance here. For larger  $D$  it decreases faster with  $m$ , which effectively compensates for the growth from the exponent in the interesting region of  $m$ . We were able to fit the meson line in Fig. 1 with formula (4), but it is impossible to fit the baryons. Moreover, the experimentally accessible range of  $m$  is not asymptotic enough to justify Eq. (4). Indeed, Eq. (4) is a direct consequence of a combinatorial problem known as *partitio numerorum*. For  $D = 1$  Eq. (4) is the asymptotic form for the number of different partitions of a positive integer  $n$  into non-negative integer components [11]. For  $D > 1$  it is a generalization to the case where the components are of  $D$  different types [9]. From the Regge formula (6) we can see that for  $m$  in the range  $1 - 2\text{GeV}$  the values of  $n$  lie between 1 and 4, hence  $n$  is not large enough to justify the asymptotic form (4). In this paper we do not pursue the dual-model track any further, and leave it with the conclusion that *it works for the meson spectrum*, and fails, in the present form, for baryons.

Is there a simple explanation of the behavior of Fig. 1? What we have to search for is a mechanism of the observed rapid increase of degrees of freedom with  $m$ . Inspired by the combinatorics of *partitio numerorum* we propose a candidate mechanism capable of explaining the result of Fig. 1 not worse than the Hagedorn model. We call it the *Compound Hadron Model*, in analogy to compound nuclei [12]. In the statistical model of nuclear reactions a compound nucleus is an object whose density of states is enormous, and is described approximately by the formula  $\rho(E^*) \simeq f(E^*) \exp(b\sqrt{E^*})$ , where  $E^*$  is the (large) excitation energy of the nucleus,  $b$  is a nucleus-dependent constant, and  $f(E^*)$  is a slowly-varying function. Let us illustrate how this formula can arise in a toy example. Suppose for simplicity that we can neglect the exclusion principle, and the system has equally-spaced single-particle excitation levels,  $\varepsilon_k = k\Delta E$ . In the ground state all particles sit in the lowest level,  $\varepsilon_0 = 0$ . The excitation energy can be composed in very many different ways. Let us express  $E^*$  in units of  $\Delta E$ , i.e.  $E^* = n \Delta E$ . We can make  $E^*$  by exciting  $n$  times the lowest (first) level,  $E^* = n\varepsilon_1$ , once the second level and  $n - 2$  times the first level,  $E^* = \varepsilon_2 + (n - 2)\varepsilon_1$ , and so on. In general,  $E^* = \varepsilon_1 k_1 + \varepsilon_2 k_2 + \dots + \varepsilon_n k_n$ , which means  $n = k_1 + 2k_2 + \dots + nk_n$ , with integer  $k_i \geq 0$ . Of course, this is nothing else but the partition problem, and the number of distinct ways this may be achieved is, for a large  $n$ , given by Eq. (5) with  $D = 1$ . The difference is, however, that now  $n$  is proportional to the excitation energy, and not the square of mass, as in Eq. (6). One can extend this combinatorial problem to account for  $D$  different types of excitations:  $n = \sum_{\alpha} k_1^{(\alpha)} + 2\sum_{\alpha} k_2^{(\alpha)} + \dots + n\sum_{\alpha} k_n^{(\alpha)}$ ,  $\alpha = 1, \dots, D$ ,  $k_i^{(\alpha)} \geq 0$ , which gives the number of partitions,  $P_D(n)$ , of Eq. (5). In addition, we may incorporate the exclusion principle

by disallowing the repetition of the same level, *i.e.* requesting that none of the numbers  $k_i^{(\alpha)}$  is repeated. This leads to the asymptotic formula [13]

$$P_D^{\text{NR}}(n) \sim n^{-3/4} \exp\left(2\pi\sqrt{\frac{D}{12}n}\right), \quad (7)$$

where “NR” indicates “no repetitions”. Note the factor of  $\frac{1}{12}$  under the square root compared to  $\frac{1}{6}$  in Eq. (5) (there are less partitions without repetitions than with repetitions), and the absence of  $D$  in the power  $n$  multiplying the exponent.

Inspired by the compound nucleus model and by the expression for  $P_D^{\text{NR}}(n)$ , we propose the following guess formula for the hadronic mass spectra, which is a direct consequence of Eq. (7):

$$\rho(m) = \frac{c\Theta(m - m_0) \exp\left(2\pi\sqrt{\frac{D}{12}\frac{(m-m_0)}{\Delta E}}\right)}{\left((m - m_0)^2 + (0.5\text{GeV})^2\right)^{5/8}}, \quad (8)$$

where  $c$  is a constant,  $m_0$  is the ground-state mass,  $D$  is the number of types of excitations, and  $\Delta E$  is the average level spacing. The asymptotic power of the function multiplying the exponent is  $-5/4$ , whereas the power  $-2$  is used in an analogous formula in compound nuclei [12]. The underlying physical picture is following: hadrons are bound objects of constituents (quarks, gluons, pions). The Fock space contains a ground state, and excitations on top of it. In the case of the compound nucleus these elementary excitations are  $1p1h$ ,  $2p2h$ ,  $3p3h$ , *etc.* states. In the case of hadrons they are formed of  $q\bar{q}$  and gluon excitations, *e.g.* for mesons we have  $q\bar{q}$ ,  $q\bar{q}g$ ,  $qq\bar{q}\bar{q}$ ,  $q\bar{q}gg$ , *etc.* Now, we form the excitation energy (hadron mass) by differently composing elementary excitations, which bring us to the above-described partition problem. It seems reasonable to take zero ground-state energy for mesons,  $m_0^{\text{mes}} = 0$ , which are excitations on top of the vacuum. For baryons we take  $m_0^{\text{bar}} = 900\text{MeV}$ , which is basically the mass of the nucleon. The quantity  $\Delta E/D$  is treated as a model parameter and is fitted to data.

The results of the compound-hadron-model fit, Eq. (8), are shown with the dashed line in Fig. 1. The curves are slightly bent down, compared to the Hagedorn-like fits (solid lines), which is caused by the square root in the exponent of Eq. (8). But the fits are no worse, or even better when the fit region is extended to  $m = 2\text{GeV}$ . Numerically, the least-square fit for  $m$  up to  $1.8\text{GeV}$  gives  $\Delta E^{\text{mes}}/D = 50\text{MeV}$  for mesons, and  $\Delta E^{\text{bar}}/D = 53\text{MeV}$  for baryons. The proximity of these numbers shows that the scales for mesons and baryons are similar. The number of types of the excitations,  $D$ , is taken to be 1. This means that we treat all levels, carrying spin, orbital, radial, or isospin quantum numbers, as separate. If we treat them as  $D$ -times degenerate, then the average spacing  $\Delta E$  increases  $D$  times, such that the dependence on  $D$  cancels. Note that in case of gluonic or  $q\bar{q}$  excitations color does not bring extra degeneracy, since all states have to be color-neutral and there is only one way in which the elementary excitations can be coupled to a color singlet. The obtained values for  $\Delta E^{\text{mes}}$  (with  $D = 1$ ) mean that the corresponding  $n$  at  $m = 1.8\text{GeV}$  is around 35 for mesons and 17 for baryons. Such  $n$  are sufficiently large to justify the use of the asymptotic formula (7).

There is an interesting effect we wish to point out. It is natural to expect that a bound hadronic system has an upper limit for the excitation energy. It is helpful to think here of bags with a finite depth, or of breaking-up flux-tubes. Thus, in constructing the Fock space for bound objects we should have a limited number of quanta to our disposal. Translating this idea into our toy combinatorial problem, we now have to ask about the number of ways of partitioning  $n$  (without repetitions) into integer components,

$n = \sum_{\alpha} k_1^{(\alpha)} + 2 \sum_{\alpha} k_2^{(\alpha)} + \dots + n \sum_{\alpha} k_n^{(\alpha)}$ , but now with  $k_{\max} \geq k_i^{(\alpha)} \geq 0$ , where  $k_{\max}$  is the maximum available number. We denote this number of partitions with limited  $k_i^{(\alpha)}$  as  $P_D^{\text{NR}}(n, k_{\max})$ . Figure 3 shows the decimal log of cumulants of this quantity for  $D = 1$ ,  $\sum_{i=1}^n P_1^{\text{NR}}(i, k_{\max})$ , plotted as a function of  $n$  for various values of  $k_{\max}$ . The case  $k_{\max} = \infty$  corresponds to partitions with unlimited  $k_i^{(\alpha)}$ . Each curve becomes flat at  $n = k_{\max}(k_{\max}+1)/2$ , which is the point where all available levels have been filled. Beyond that point we cannot form bound states any more. We term this effect *combinatorial saturation*. It means that in a compound-hadron model beyond a certain mass  $m$  no more light-quark resonances can be formed. For instance, our toy model with  $D = 1$  and  $k_{\max} = 12$  predicts that this happens at  $n = 78$ . With  $\Delta E$  obtained in the fits, this leads, according to the formula  $m = m_0 + n\Delta E$ , to a maximum mass of  $\sim 4\text{GeV}$  for light-quark mesons and  $\sim 5\text{GeV}$  for light-quark baryons. We can see in Fig. 3 that before reaching the combinatorial saturation, the curves with a given  $k_{\max}$  depart from the unlimited-partition curve,  $k_{\max} = \infty$ . It turns out that the  $k_{\max} = \infty$  curve is very close to the asymptotic (large  $n$ ) curve, described by formula (7), even for low values of  $n$ . This is why Eq. (8) can be used even at low values of  $m$ . We may thus speculate that the departure of data from the dashed lines in Fig. 1 around  $m = 2\text{GeV}$ , similar to the behavior of Fig. 3, is a signal of the combinatorial saturation. More complete data in that region and above would definitely help to check if this is really the case. We note that Bisudová, Burakovsky and Goldman [14] obtain similar limits for mass of light hadrons from non-linear Regge trajectories.

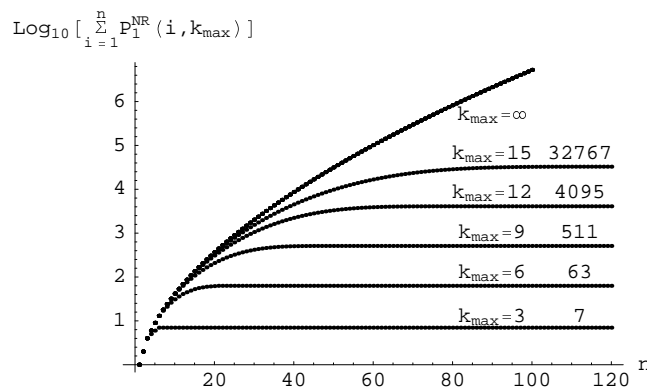


FIG. 3. Cumulants of  $P_1^{\text{NR}}(i, k_{\max})$  for various values of  $k_{\max}$ . The integers at the right ends of the curves are the saturation values of the cumulants.

Let us summarize the basics of combinatorics behind the formation of the hadronic spectra. As we increase the mass, more elementary excitations may participate in the resonance-formation process, in other words *more degrees of freedom* open up. This, via partitio numerorum, leads to an exponential growth of the number possibilities of forming excited states. Beyond a certain mass there are no more elementary excitations available, and we reach the combinatorial saturation in the number of hadronic states. Clearly, the real life of hadronic physics is more complicated than the toy model discussed above. However, the result may be robust and not care about the details, for example whether the levels are equally spaced, as assumed here, or not. The success of the statistical model in nuclear physics shows that details are not needed to understand gross features system with many degrees of freedom.

Finally, let us comment on the different power of  $m$  in the exponent of Eq. (8) and the Hagedorn hypothesis, Eq. (1). It is not a paradox, since our simple assumptions in the language of quark and gluon (or pion) excitations, are completely different than the assumptions of the Statistical Bootstrap Model, where self-similarity of fireballs is the key

ingredient. The thermodynamical implications of the two models are different: whereas in the Hagedorn model the temperature  $T$  of a hadronic system cannot be larger than the Hagedorn temperature, in the Compound Hadron Model there is no formal limit on  $T$ . We stress that this is of formal significance only, since deconfinement is expected to occur at temperatures of the order of  $\sim 150\text{MeV}$ , dissolving hadron resonances into the quark-gluon plasma.

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