Non-axisymmetry of blood flow in the presence of an external magnetic field B_0: evaluation of the shear stresses at the vessel wall

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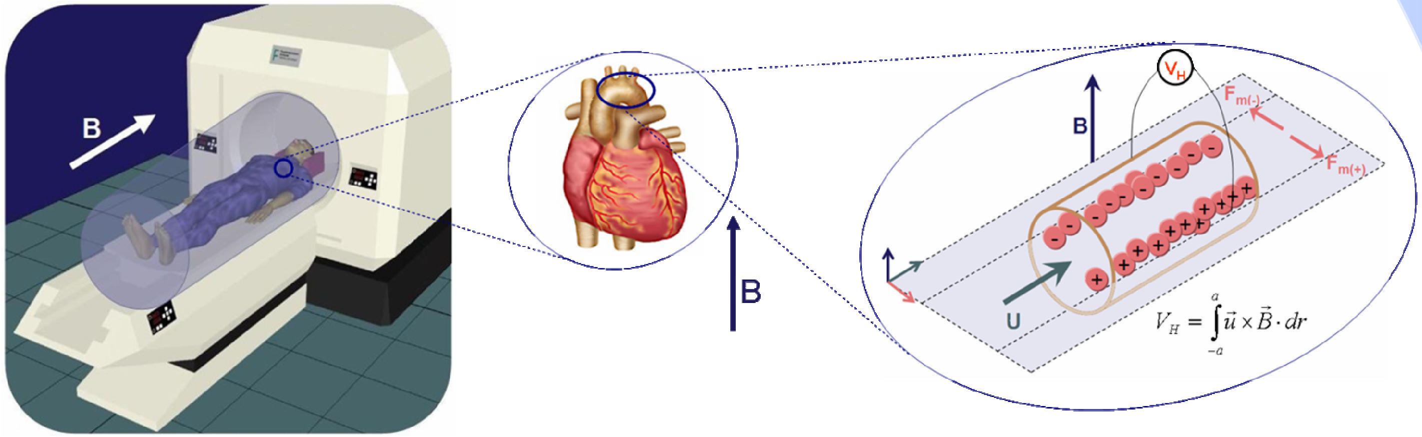
Hall effect and Lorentz force

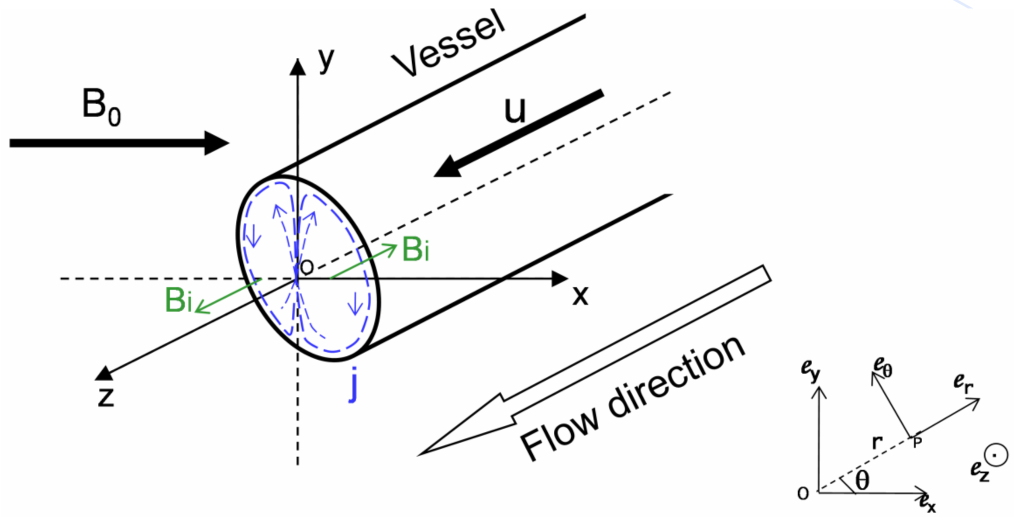
Hall effect : *The positively and negatively charged blood particles, flowing transversally to the B_0 field, get deflected by the Lorentz force in opposite directions; some electric potentials (and electric fields) are thus induced and the Lorentz force opposes to the flow; if a return path exists for the current (« current loop »), a magnetic field is induced*

Lorentz force : *$j \wedge B_0$, where j is the current density. If the induced fields are neglected, j may be obtained by Ohm's law:* $\vec{j} = \sigma(\vec{u} \wedge \vec{B}_0)$

Example: MRI examinations



Induced potentials and fields



Non-conducting walls :

- → *Induced currents are captured*
- → *Current loops inside the vessel*
- → *On the return path, compensating Lorentz force*
- → *Flow retardation is lower*

Neglecting the induced fields also overestimates the flow retardation (current loops neglected)

- *Classical Poiseuille flow: u(r) and $\tau = \eta\ \partial u/\partial r$*

  MHD flow of blood : $u(r, \theta)$ and $\tau = ?$

Motivation of the study

- *ECG gated magnetic resonance imaging (cardiac MRI): the MHD induced potential gets superimposed on the recorded ECG signal and this impedes correct synchronization.*

  → *Prediction of MHD perturbation on ECG signal in order to be able:*
  - to eliminate this artefact,
  - or, on the contrary, to use it (elevation of the T-wave) as the synchronization tool,
  - or to use it as a non-invasive measure of the cardiac output!

  (Abi-Abdallah's papers, 2007-2009; Frauenrath et al., J; Mag.Res.Imag., 2012; Kyriakou et al., Physiol. Meas., 2012)

- *MRI measurements of aortic pulse wave velocity (indicator of arterial wall stiffness) in some cardiovascular disease*

  (Ibrahim et al., J. Card. Mag. Resonance, 2010; Markl et al., Magn. Res. Medicine, 2010)

- *Energy harvesting from the pressure-driven deformation of an artery by the principle of magneto-hydrodynamics (long term objective = to design some micro-generators using intracorporeal energy, that could avoid the replacement of the batteries of medical implants).*

  (Pfenniger et al., Med. Engin. Physics, 2013; Med. Biol. Engin. Comput., 2013)

Motivation of the study (2)

- *Tissue engineering*

  → *Use of magneto-responsive particles to improve cellular invasion and adhesion in the scaffolds*

  (Castro and Mano, Jour. Biomed. Nanotech., 2013
  Xu et al., Jour. Biosciences Bioengin., 2008, and many others …)

- *Magnetic drug transport and targetting*

  → *magnetic particles containing or coated with therapeutics are concentrated to sites of disease by applied magnetic fields.*

  (Sensenig et al., Nanomedecine, 2012; Nacev et al., Nanomedecine, 2010; and many others)

- *Mechano-transduction studies (regenerative medicine, stem cells, …)*

  (Santos et al., Trends in Biotechnology, 2015; …)

- *Risk assessment for the vessel wall*

  Plaque rupture, aneurysm, cell attachment and /or transmigration, …

  (Boussel et al., Magn. Reson. Med., 2009; …)

Blood flow in magnetic field

General equations

N-S equa., including Lorentz force

$$\rho_f\left(\frac{\partial\vec{u}}{\partial t}+(\vec{u}.\vec{\nabla})\vec{u}\right)=-\vec{\nabla}P+\vec{j}\wedge\vec{B}+\mu_f\Delta\vec{u}+\rho_f\vec{g}$$

Induced electric field neglected

Ohm's law

$$\vec{j}=\sigma(\vec{E}+\vec{u}\wedge\vec{B})$$

(E_{ext}= 0)

Induced electric field not neglected

M4 + quasi sta. approx

$$\vec{\nabla}\wedge\vec{B}=\mu_m\vec{j}$$

Maxwell equations

$$(M1):\vec{\nabla}.\vec{E}=\frac{q}{\varepsilon}$$

$$(M2):\vec{\nabla}\wedge\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$(M3):\vec{\nabla}.\vec{B}=0$$

$$(M4):\vec{\nabla}\wedge\vec{B}=\mu_m(\vec{j}+\varepsilon\frac{\partial\vec{E}}{\partial t})$$

q = charge density (C/m³)
j = current density (A/m²)
μ_m = magnetic permeability (H/m)
ε= electric permittivity (F/m)
σ = blood conductivity (S/m)

[Ohm's law + M4(q.s.a.) +M2] gives induction equation

$$\frac{\partial\vec{B}}{\partial t}=\vec{\nabla}\wedge(\vec{u}\wedge\vec{B})+\frac{1}{\sigma\mu_m}\Delta\vec{B}$$

Existing analytical solutions

References	Geometry	Pressure gradient	Walls	Induced fields
Hartmann, Mat. Fys. Med., 1937	Parallel plates	Constant	Rigid, non conducting	Neglected
Vardanyan, Biofizika, 1973	Cylindrical pipe	Constant	Rigid, non conducting	Neglected
Gold, JFM, 1962	Cylindrical pipe	Constant	Rigid, non conducting	Not neglected
Sud et al.,Stud. Biophys., 1974	Cylindrical pipe	Harmonic	Rigid, non conducting	Neglected
Abi-Abdallah et al., CMBBE, 2009	Cylindrical pipe	Physiologic	Rigid, non conducting	Neglected
Drochon, E.P.J.-A.P.,2016	Cylindrical pipe	Harmonic	Deformable, non conduct.	Neglected

Gold (JFM, 1962)

constant pressure gradient in a cylindrical rigid pipe, with transverse magnetic field

Newtonian fluid; non conducting walls; induced magnetic field not neglected

Objective: Velocity and induced electrical potential

Modified Navier-Stokes equation + induction equation

+ boundary conditions: $u_z(R,\theta) = 0$ and $B_z(R, \theta) = 0$

→ Solved for: Velocity ($u_r=0$, $u_\theta=0$, $u_z(r,\theta)$)
Induced magnetic field ($B_r=B_0\cos\theta$, $B_\theta=-B_0\sin\theta$, $B_z(r,\theta)=B_I$)

→ Induced current density (M-A law): $$j_{Ir} = \frac{1}{\mu_m r}\frac{\partial B_I}{\partial \theta};\; j_{I\theta} = -\frac{1}{\mu_m}\frac{\partial B_I}{\partial r};\; j_{Iz} = 0$$

→ Induced electric field (Ohm's law): $$E_{Ir} = \frac{1}{\sigma} j_{Ir} + u_z B_\theta;\; E_{I\theta} = \frac{1}{\sigma} j_{I\vartheta} - u_z B_r;\; E_{Iz} = 0$$

→ Maximal induced electric potential: $$V = 2\int_0^R E_{Ir}(r, \frac{\pi}{2})dr$$

Gold's velocity profiles

$$\tilde{U}(\tilde{r},\theta) = \frac{G}{2H_a}\left[E_1(\tilde{r},\theta)\,A_1(\tilde{r},\theta) + E_2(\tilde{r},\theta)\,A_2(\tilde{r},\theta)\right]$$

With :

$$E_1(\tilde{r},\theta) = e^{-\frac{H_a}{2}\tilde{r}\cos\theta} \qquad E_2(\tilde{r},\theta) = e^{\frac{H_a}{2}\tilde{r}\cos\theta}$$

$$A_1(\tilde{r},\theta) = \alpha_0 I_0(\frac{H_a}{2}\tilde{r}) + \sum_{n=1}^{\infty} 2\alpha_n I_n\left(\frac{H_a}{2}\tilde{r}\right)\cos(n\theta)$$

$$A_2(\tilde{r},\theta) = \alpha_0 I_0(\frac{H_a}{2}\tilde{r}) + \sum_{n=1}^{\infty} 2\,(-1)^n \alpha_n I_n\left(\frac{H_a}{2}\tilde{r}\right)\cos(n\theta)$$

$$\alpha_0 = \frac{I_0'(\frac{H_a}{2})}{I_0(\frac{H_a}{2})}; \quad \alpha_n = \frac{I_n'(\frac{H_a}{2})}{I_n(\frac{H_a}{2})}$$

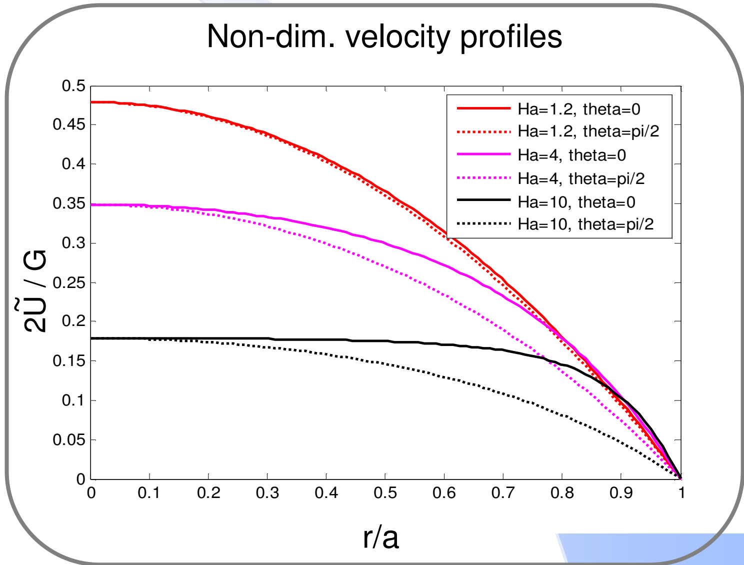
The functions I_n are the n^{th} order modified Bessel functions of the first kind.

$$H_a = B_0 R\sqrt{\frac{\sigma}{\eta}}$$

is the Hartmann number

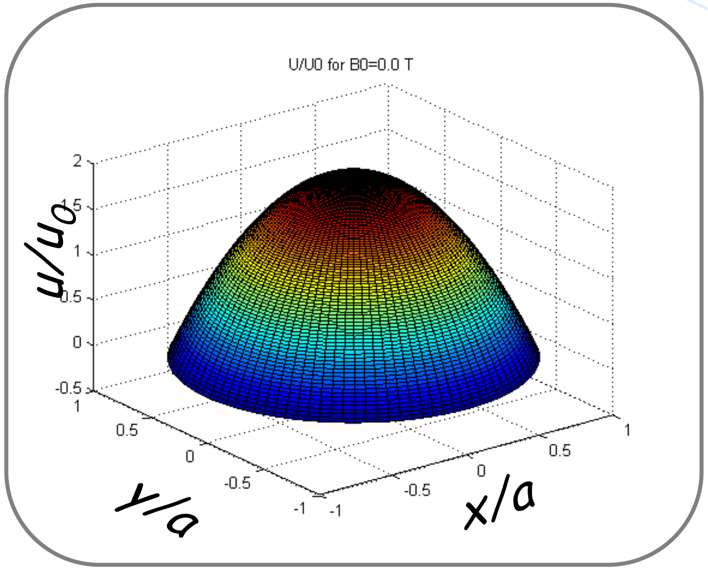
$$\tilde{U} = \frac{u}{u_0}$$

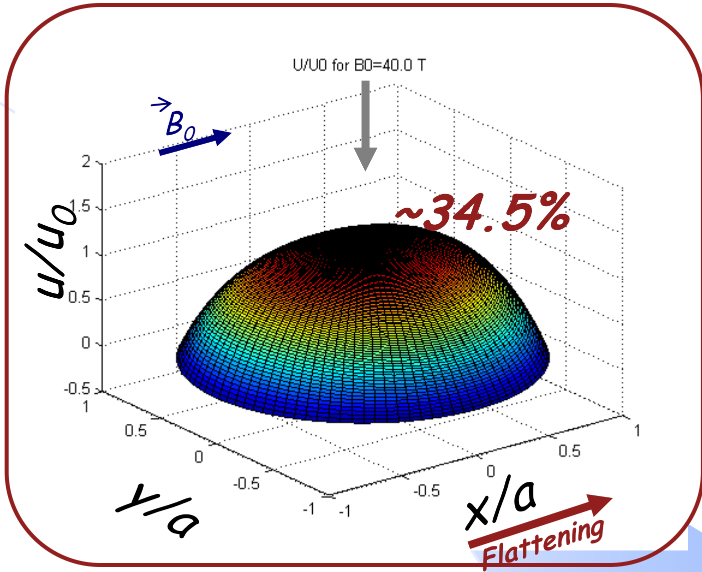
$$G = -\frac{R^2}{\mu_f u_0}\frac{\partial P}{\partial z}$$



- *When H_a ↗, velocity is reduced and velocity profile is flattened*
- *If H_a =0, Poiseuille profile:* $\frac{2\tilde{U}}{G}(r=0) = 0.5$

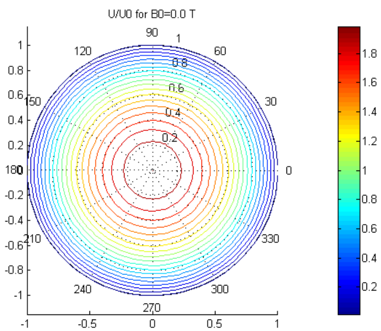
Gold's solution: velocity

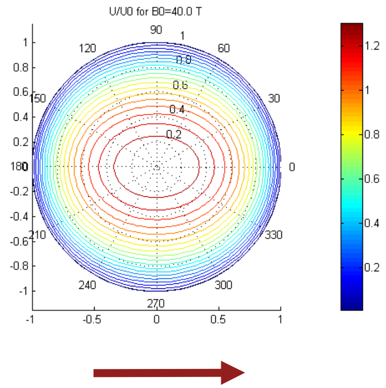




$B_0 = 0T$

$B_0 = 40T$

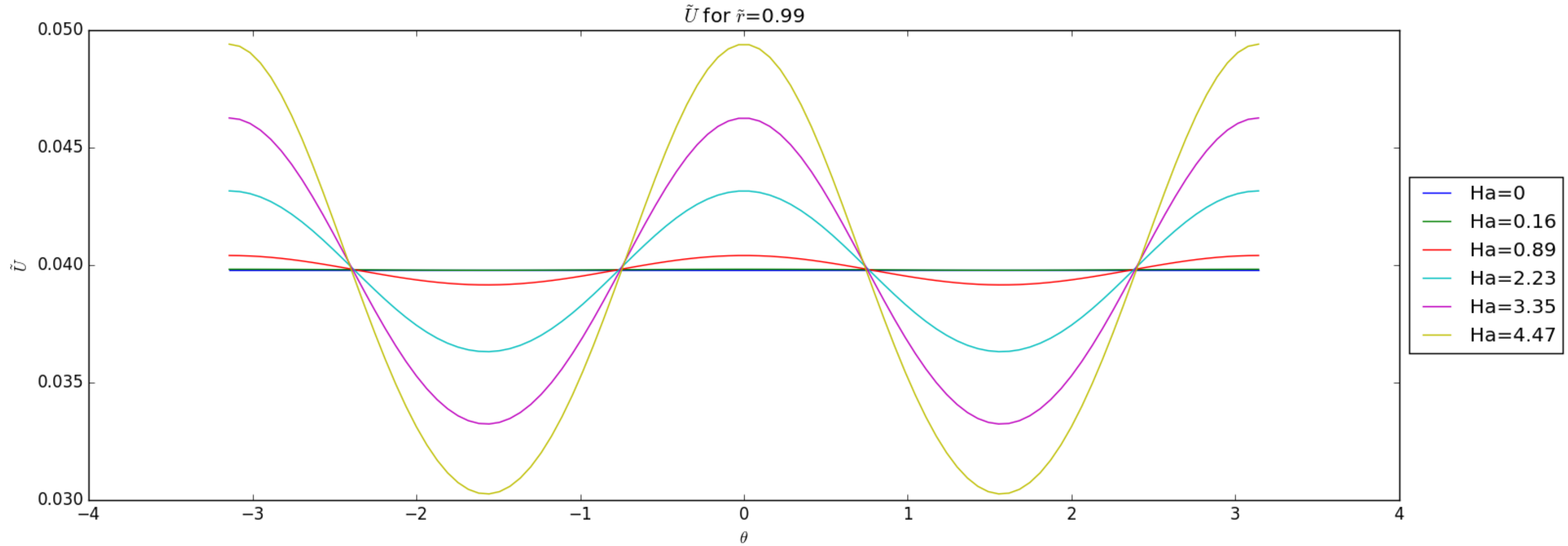




- *Case H_a = 4.47 (B_0 = 40T), iso-velocity lines (more or less tightened) give a representation of the gradients*

Dependence of the velocity on θ

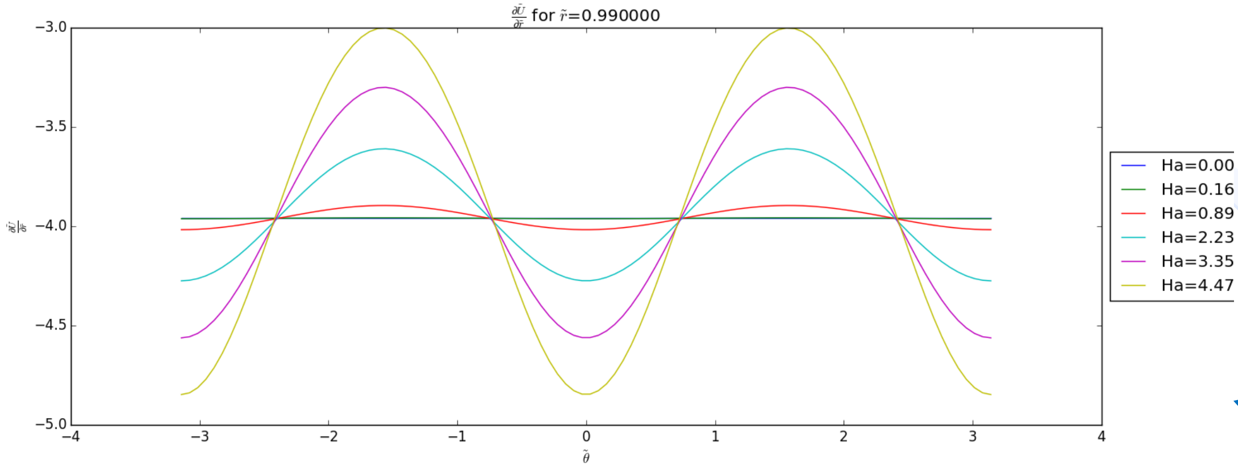
(Close to the wall: r/R = 0.99)



- *Velocity maximal in $\theta = 0$, $\theta = \pi$, … (stretching parallel to the direction of B_0)*
- *The dependence on θ increases with H_a*

Velocity gradients

- In the radial direction (at r/R = 0.99) : $\dfrac{\partial \tilde{U}}{\partial \tilde{r}}$



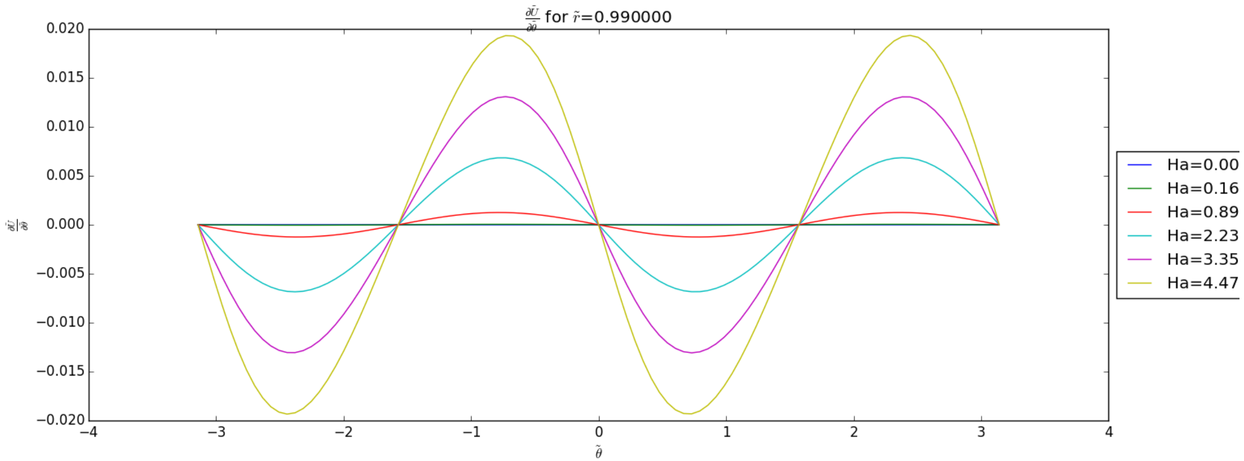
Negative values, because U = 0 at the wall

Poiseuille value (-4), when $H_a = 0$

Max. value of the gradient in $\theta = 0$ or $\theta = \pi$, …

Dependence on θ increases when H_a increases

- In the azimuthal direction (at r/R = 0.99): $\dfrac{\partial \tilde{U}}{\partial \theta}$



No dependence on θ when $H_a = 0$!

Max. value of the gradient in $\theta = \pi/4$ or $\theta = 3\pi/4$, …

Shear Stresses ?

- *We demonstrate that:*

$$\tilde{\tau}_{rz} = \frac{\partial \tilde{U}}{\partial \tilde{r}} >> \tilde{\tau}_{\theta z} = \frac{1}{\tilde{r}} \frac{\partial \tilde{U}}{\partial \theta}$$

Both quantities depend upon θ, but this dependence may be considered negligible for low values of B_0

- *Integration of Navier-Stokes equation over a cross-section of the vessel*

(In Drochon et al., Appl. Math., 2016)

$$\begin{cases} \iint_A (\Delta u_z) dA = \dfrac{\pi R^2}{\mu_f} \left(\dfrac{\partial P}{\partial z}\right) \\ \iint_A (\Delta B_I) dA = 0 \end{cases} \longrightarrow \mu_f R \int_0^{2\pi} \frac{\partial u_z}{\partial r}(R, \theta) d\theta = \pi R^2 \left(\frac{\partial P}{\partial z}\right)$$

(Equilibrium of forces exerted on an elementary volume of the vessel, of length 1, in case of MHD flow)

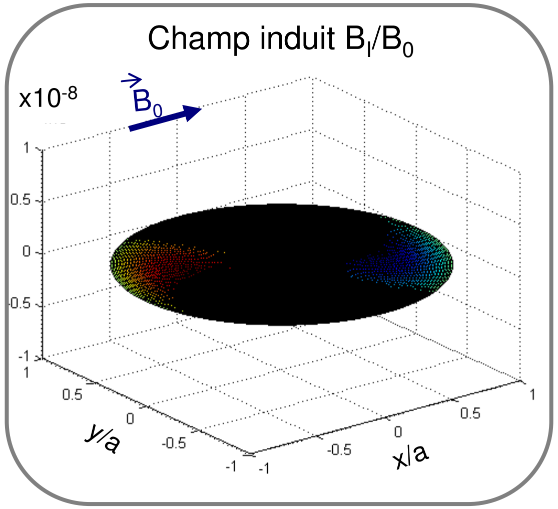
Other publications associated with this work

- *Abi-Abdallah et al., Eur. Phys. Jour. –Appl. Phys., 2009*
- *Drochon et al., Jour. Appl. Math. Phys., accepted, 2017*

Induced magnetic field

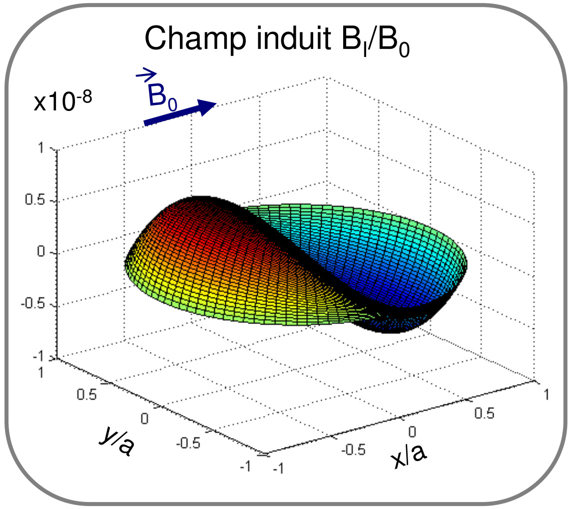
- *Induced field value is not proportional to B_0 : when $B_0 \nearrow$, the flow is retarded further, thus reducing the inductions*
- *Charge separation occurs along Oy (perpendicular to the flow and to B_0)*

$H_a = 0.16$



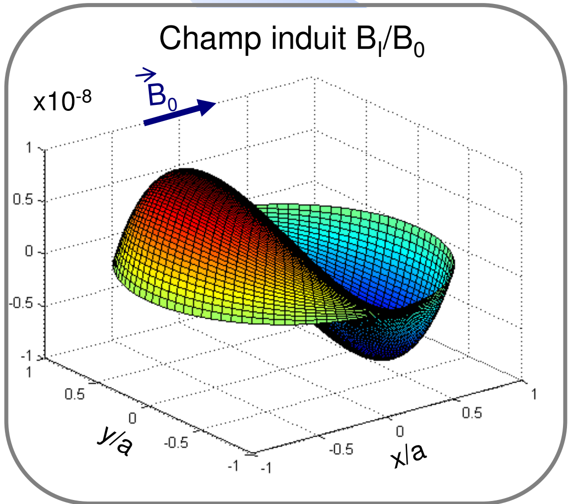
$B_0 = 1.5T$

$H_a = 2.23$



$B_0 = 20T$

$H_a = 4.47$

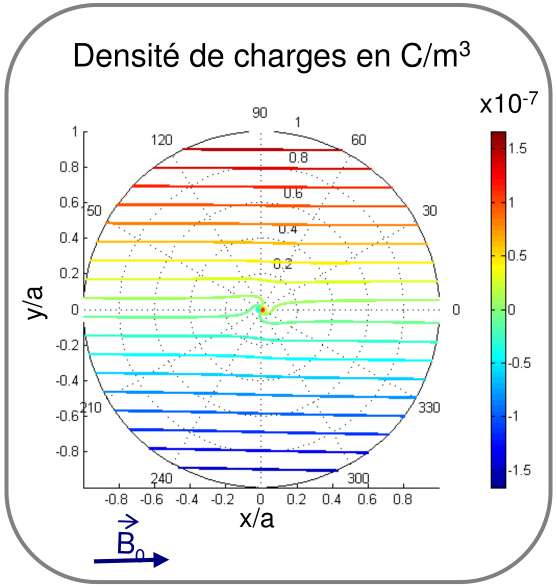


$B_0 = 40T$

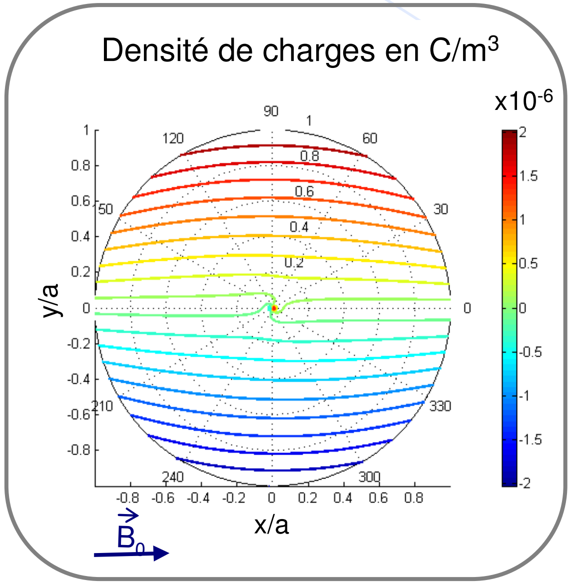
$B_I < 4.10^{-7}$ T even for B_0=40T

(earth's magnetic field = 5.10^{-5}T)

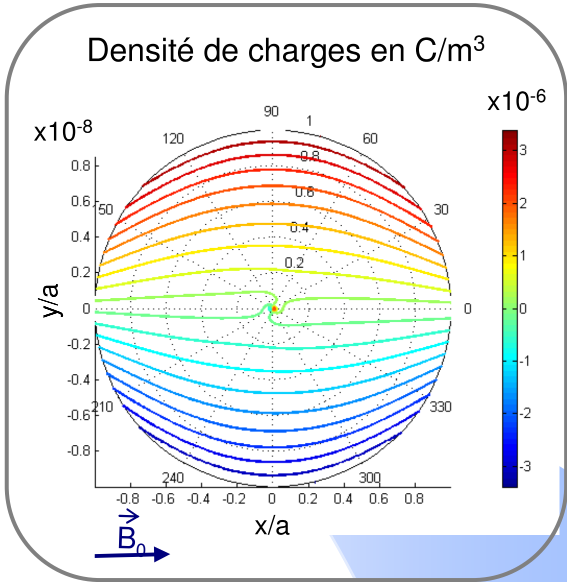
Charge density



$B_0 = 1.5T$



$B_0 = 20T$



$B_0 = 40T$

- *Separation is better emphasized when B_0 ↗*
- ***E_I** oriented along (-$\mathbf{e}_y$), in opposite direction to the main current*

Averaged values over a cross-section of the vessel

(Case of insulating vessel walls and calculated inductions)

$$\iint_A (\vec{j} \wedge \vec{B}).\vec{e}_z \, dA = 0 \quad \text{and} \quad \iint_A [rot(\vec{u} \wedge \vec{B})].\vec{e}_z \, dA = 0$$

→ *So that N.S. equ. and induction equ. become uncoupled :*

$$\begin{cases} \iint_A (\Delta u_z) dA = \dfrac{\pi R^2}{\mu_f}\left(\dfrac{\partial P}{\partial z}\right) \\ \iint_A (\Delta B_I) dA = 0 \end{cases}$$

← *Does not depend on B_0, whereas the mean velocity does*

Can be transformed as:

$$\int_0^{2\pi} \frac{\partial B_I}{\partial r}(R,\theta) d\theta = 0 \text{ and thus } \int_0^{2\pi} j_\theta(R,\theta) d\theta = 0$$

- *The induced currents circulating in closed loops compensate each other exactly*

Non-dimensional j_θ